

Binomial Symbols and Prime Moduli

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Abstract. We try to improve a problem asked in an Indian Math Olympiad. We give a brief overview of the work done in [4] and [5] where the authors have found a periodic sequence and the length of its period, all inspired from an Olympiad problem. The main goal of the paper is to improve the main result in [4].

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1. MOTIVATION

The motivation behind this work was an Indian Olympiad problem mentioned in [4]. Saikia and Vogrinc in [4] have proved the following result.

Theorem 1.1. *A natural number $p > 1$ is a prime if and only if $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative n , where $\binom{n}{p}$ is the number of different ways in which we can choose p out of n elements and $\lfloor x \rfloor$ is the greatest integer not exceeding the real number x .*

We give three different proofs of the above result in [4]. For the sake of completeness we give below a proof.

Proof. First assume that p is prime. Now we consider n as $n = ap + b$ where a is a non-negative integer and b an integer $0 \leq b < p$. Obviously,

$$\lfloor \frac{n}{p} \rfloor = \lfloor \frac{ap+b}{p} \rfloor \equiv a \pmod{p}.$$

Now let us calculate $\binom{n}{p} \pmod{p}$.

$$\begin{aligned} \binom{n}{p} &= \binom{ap+b}{p} \\ &= \frac{(ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot ap \cdot (ap-1) \cdots (ap+b-p+1)}{p \cdot (p-1) \cdots 2 \cdot 1} \\ &= \frac{a \cdot (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdots (ap+b-p+1)}{(p-1) \cdot (p-2) \cdots 2 \cdot 1} \end{aligned}$$

We denote this number by X .

We have $X \equiv c \pmod{p}$ for some $0 \leq c < p$. Consequently taking modulo p , we have

$$c(p-1)! = X(p-1)! = a(ap+b) \cdots (ap+1)(ap-1) \cdots (ap+b-p+1)$$

All the numbers $ap+b, \dots, ap+b+1-p$ (other than ap) are relatively prime to p and obviously none differ more than p so they make a reduced residue system modulo p , meaning we have mod p ,

$$(p-1)! = (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdot (ap+b-p+1)$$

both sides of the equation being relatively prime to p so we can deduce $X \equiv c \equiv a \pmod{p}$. And finally $\binom{n}{p} \equiv X \equiv a \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$.

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To complete the other part of the theorem we must construct a counterexample for every composite number p . If p is composite we can consider it as $q^x \cdot k$ where q is some prime factor of p , x its exponent and k the part of p that is relatively prime to q (x and k cannot be simultaneously 1 or p is prime). We can obtain a counterexample by taking $n = p + q = q^x k + q$ will make a counter example. We have:

$$\binom{p+q}{p} = \binom{p+q}{q} = \frac{(q^x k + q)(q^x k + q - 1) \dots (q^x k + 1)}{q!}$$

Which after simplifying the fraction equals: $(q^{x-1}k + 1) \frac{(q^x k + q - 1) \dots (q^x k + 1)}{(q-1)!}$. Similarly as above we have $(q^x k + q - 1) \dots (q^x k + 1) = (q-1)! \neq 0$ modulo q^x therefore,

$$\frac{(q^x k + q - 1) \dots (q^x k + 1)}{(q-1)!} \equiv 1 \pmod{q^x}$$

and

$$\binom{p+q}{p} \equiv q^{x-1}k + 1 \pmod{q^x}.$$

On the other hand obviously,

$$\lfloor \frac{q^x k + q}{q^x k} \rfloor \equiv 0 \pmod{q^x}.$$

And since $q^{x-1}k + 1$ can never be equal to 0 modulo q^x we see that

$$\binom{p+q}{p} \not\equiv \lfloor \frac{p+q}{p} \rfloor \pmod{q^x}$$

consequently also incongruent modulo $p = q^x k$. □

Remark 1.2. Here we would like to comment that by taking q as the minimal prime factor of p and using the same method as above we can simplify the proof even more. We can then compare $\lfloor \frac{p+q}{p} \rfloor$ and $\binom{p+q}{p}$ directly modulo $p = q^x k$ and not q^x .

Remark 1.3. Instead of looking modulo p , we can look at higher powers of p , or we can look at the n -th Fibonacci prime and so on. However, initial investigations by the authors suggest that finding a congruence relation in those cases becomes more difficult.

This theorem is the motivation behind the following two theorems.

Theorem 1.4. The sequence $a_n = \binom{m}{x} \pmod{m}$ is periodic, where $x, m \in \mathbb{N}$.

The proof based on mathematical induction can be found in [2] and [5]. We present below a slightly modified account.

Proof. If $x = 1$ the sequence is obviously periodic for any modulo m .

Now we assume that the sequence is periodic for a fixed x and arbitrary m . We note that

$$\binom{n}{x+1} = \sum_{i=1}^{n-1} \binom{i}{x}.$$

Let k be the length of a period of sequence $a_n = \binom{n}{x} \pmod{m}$, meaning $\binom{n+k}{x} \equiv \binom{n}{x} \pmod{m}$.

Therefore $\sum_{i=1}^k \binom{i}{x} \equiv c \pmod{m}$ for some c and consequently $\sum_{i=n+1}^{n+mk} \binom{i}{x} = mc = 0 \pmod{m}$ for every integer n . All that is now required is another calculation (the second equality from the right is modulo m):

$$\binom{n+mk}{x+1} = \sum_{i=1}^{n+mk-1} \binom{i}{x} = \sum_{i=1}^{n-1} \binom{i}{x} + \sum_{i=n}^{n+mk-1} \binom{i}{x} = \sum_{i=1}^{n-1} \binom{i}{x} = \binom{n}{x+1}$$

This now shows that sequence $b_n = \binom{n}{x+1} \pmod{m}$ is also periodic for every modulo m which completes the induction and yields the desired result. □

Remark 1.5. Because of the upper result we know that a sequence $a_n = \binom{n}{m} \pmod{m}$ is also periodical. And a sequence $c_n = \lfloor \frac{n}{m} \rfloor \pmod{m}$ is obviously periodical. This combined with the above yields that for a composite modulo m there exist infinitely many natural numbers n such that $a_n \neq c_n \pmod{m}$.

The above theorem states that for every m the sequence $a_n = \binom{n}{m} \pmod{m}$ is periodic. The next most natural question to ask is, what is the minimal length of the period? Which gives us,

Theorem 1.6. For a natural number $m = \prod_{i=1}^k p_i^{b_i}$, the sequence $a_n = \binom{n}{m} \pmod{m}$ has a period of minimal length,

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}$$

The proof given below is the one given by the authors in [5].

Proof. A sequence $a_n = \binom{n}{m} \pmod{m}$ where $m = \prod_{i=1}^k p_i^{b_i}$ starts with m zeroes (we start with a_0). Now let us see when is the next time we have m consecutive zeroes in the sequence a_n . We assume this happens at some natural number n , that is

$$\binom{n}{m} \equiv \binom{n+1}{m} \equiv \dots \equiv \binom{n+m-1}{m} \equiv 0 \pmod{m}.$$

Let p be a prime dividing m and b be it's exponent in the prime factorisation of m . We have $\binom{n+i}{m} \equiv 0 \pmod{p^b}$ for $0 \leq i < m$.

Obviously the exponent of p in prime factorisation of $m!$ is

$$\vartheta_p(m) = \sum_{i=1}^{\infty} \lfloor \frac{m}{p^i} \rfloor = \sum_{i=1}^k \lfloor \frac{m}{p^i} \rfloor,$$

where k is the last summand different to zero and $k = \lfloor \log_p m \rfloor$.

Among numbers $n+1, n+2, \dots, n+m$ there exist one that is divisible by p^k (there are m consecutive numbers and $m \geq p^k$). We denote this number by x . We have,

$$\binom{x-1}{m} = \frac{(x-1)(x-2)\dots(x-m)}{m!}.$$

Since we have $-(x-i) \equiv i \pmod{p^j}$ for all $1 \leq i < m$ and $1 \leq j \leq k$, so there are same number of numbers divisible by p^j in $(x-1)(x-2)\dots(x-m)$ as in $m!$ for $1 \leq j \leq k$.

On the other hand we have $\binom{x-1}{m} \equiv 0 \pmod{p^b}$ (since $x-1$ is one of the numbers $n, n+1, \dots, n+m-1$). Of m consecutive integers obviously only one can be divisible by p^j if $j < k$. Therefore if we want the numerator of $\binom{x-1}{m}$ to have exponent of p for b larger than the denominator (that is in order to have $\binom{x-1}{m} \equiv 0 \pmod{p^b}$) we need one of the numbers of the nominator to be divisible by $p^{\lfloor \log_p m \rfloor + a}$. Denote this number by y .

We assume $y \neq n$. Than either $y+m$ (if $y < n$) or $y-1$ (if $y > n$) are in the set $n, n+1, \dots, n+m-1$. This means that

$$\binom{y+m}{m} \equiv 0 \pmod{p^b}.$$

(The other case is very similar and uses the same argument.)

However that is imposible since $y \equiv 0 \pmod{p^{k+1}}$ meaning the exponent of p in prime factorisation of $(y+m)(y+m-1)\dots(y+1)$ is the same as in prime factorisation of $m!$ or in other words that $\binom{y+m}{m}$ is relatively prime to p . We reached a contradiction which means $y = n$.

The same argument will work for any arbitrary prime number dividing m . That means for every prime number p dividing m (infact $p^b | m$) we need n to be divisible by $p^{\lfloor \log_p m \rfloor + b}$, therefore the length of the period of the sequence, a_n must be a multiple of the number $\prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}$.

All that remains is to show that this infact is the lenght of the period. We need to prove that for every natural number n we have

$$\binom{n}{m} \equiv \binom{n+l(m)}{m} \pmod{m},$$

where

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

Because of some basic properties of congruences ($a \equiv b \pmod{m}$ equivalent to $ax \equiv bx \pmod{m}$) if $\gcd(m, x) = 1$), it is enough to show that,

$$\frac{\prod_{i=0}^{m-1} (n-i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}} \equiv \frac{\prod_{i=0}^{m-1} (n+l(m)-i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}} \pmod{m}.$$

Among the numbers $n, n-1 \dots n-m+1$ there are atleast $\lfloor \frac{n}{p^l} \rfloor$ that are diviiable by p^l for every positive integer l and any prime divisor p of m .

This is because

$$\prod_{i=0}^{m-1} (n-i) = \frac{n!}{(n-m)!}$$

and

$$\vartheta_p(a+b) \geq \vartheta_p(a) + \vartheta_p(b).$$

The fraction $\frac{\prod_{i=0}^{m-1} (n-i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}}$ can therefore be simplified in such a way that no number of the product $\prod_{i=0}^{m-1} (n-i)$ is divided by p on exponent greater than $\lfloor \log_p m \rfloor$.

In other words the fraction $\frac{\prod_{i=0}^{m-1} (n-i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}}$ can be simplified as $\prod_{i=0}^{m-1} \frac{n-i}{\prod_{j=1}^k p_j^{c_j}}$ where for each j, i we have $n-i$ divisable by $p_j^{c_j}$ and $c_j \leq \lfloor \log_{p_j} m \rfloor$ (each factor is an integer).

But then since for every j we have $\prod_{j=1}^k p_j^{c_j}$ divides $\prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor}$ and since $m \cdot \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor} = l(m)$ we have for every $i \pmod{m}$,

$$\frac{n+l(m)-i}{\prod_{j=1}^k p_j^{c_j}} = \frac{n-i}{\prod_{j=1}^k p_j^{c_j}} + \frac{l(m)}{\prod_{j=1}^k p_j^{c_j}} = \frac{n-i}{\prod_{j=1}^k p_j^{c_j}} + t \cdot m = \frac{n-i}{\prod_{j=1}^k p_j^{c_j}}$$

This completes the result and hence the length of the minimal period of the sequence, a_n is

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

□

Remark 1.7. If we define $\binom{n}{m}$ also for negative integers n , as $\binom{n}{m} = \frac{\prod_{i=0}^{m-1} (n-i)}{m!}$ we can adopt the minimal period length formula for all integers (we can prove in exactly the same way that $\binom{n+l(m)}{m} = \binom{n}{m}$ for every integer n).

The above theorem gives us very easily the following two corollaries:

Corollary 1.8. For every positive integer $m = \prod_{i=1}^k p_i^{b_i}$ we have $m^2 | l(m)$.

Corollary 1.9. m has only one prime factor ($m = p^b$ where p is prime) if and only if $l(m) = m^2$.

We donot prove the corollaries here, as it is quite evident that they follow from the previous theorem.

2. MAIN RESULT

Before we prove our main result, we shall state and prove two lemmas.

Lemma 2.1. *Let n be relatively prime to m . Then,*

$$\binom{n}{m} \equiv \binom{n-1}{m} \pmod{m}.$$

Proof. Note that if n is relatively prime to m than so is $n - m$. We have

$$\binom{n}{m} = \binom{n-1}{m} \cdot \frac{n}{n-m} \pmod{m}$$

which is equivalent to

$$(n-m) \cdot \binom{n}{m} = n \cdot \binom{n-1}{m} \pmod{m}$$

which is further equivalent to

$$\binom{n}{m} = \binom{n-1}{m} \pmod{m}$$

because $n = n - m \pmod{m}$ and both are relatively prime to m . □

Lemma 2.2. *Let m be even. Then for every integer k we have,*

$$\binom{m+k}{m} \equiv \binom{l(m)-1-k}{m} \pmod{m}.$$

Proof. We have

$$\binom{l(m)-1-k}{m} = \frac{(l(m)-1-k)(l(m)-1-k-1)\dots(l(m)-1-m-k+1)}{m!}$$

and because there are an even number (m) of factors we can multiply each one by -1 and still have the same number. So,

$$\binom{l(m)-1-k}{m} = \frac{(k+1-l(m))(k+2-l(m))\dots(k+m-l(m))}{m!}$$

which is precisely $\binom{k+m-l(m)}{m}$ and is by the previous theorem equal to $\binom{m+k}{m} \pmod{m}$. □

We state without proof the following famous theorem in Number Theory

Theorem 2.3. (P. G. L. Dirichlet, 1837) *If a and b are relatively prime positive integers, then the arithmetic progression*

$$a, a + b, a + 2b, a + 3b, \dots$$

contains infinitely many primes.

Now, we are in a position to state and proof a result which is stronger than **Theorem 1.1**,

Theorem 2.4. *A natural number $p > 1$ is a prime if and only if $\binom{q}{p} - \lfloor \frac{q}{p} \rfloor$ is divisible by p for every prime q , where $\binom{q}{p}$ is the number of different ways in which we can choose p out of q elements and $\lfloor x \rfloor$ is the greatest integer not exceeding the real number x .*

Proof. If p is prime than the result follows from **Theorem 1.1**. To prove the if only part we just have to construct a counterexample for every composite p .

We assume p is an even composite number. We have $\binom{p}{p} = 1 \pmod{p}$ and by **Lemma 2.1** we have $\binom{l(p)-1}{p} = \binom{p}{p} = 1 \pmod{p}$. And by **Theorem 1.4** we have $\binom{k \cdot l(p)-1}{p} = 1$ for every natural k .

On the other hand because $p^2 | l(p)$ we have $\lfloor \frac{k \cdot l(p)-1}{p} \rfloor = \lfloor \frac{Ap^2-1}{p} \rfloor = -1 \pmod{p}$. Therefore $\binom{n}{p} \neq \lfloor \frac{n}{p} \rfloor \pmod{p}$ for every integer n in the sequence $b_k = k \cdot l(p) - 1$ and since 1 and $l(p)$ are relatively prime we have a prime number in this sequence by **Theorem 2.3**. Therefore there exists a prime number q such that $\binom{q}{p} \neq \lfloor \frac{q}{p} \rfloor \pmod{p}$.

Now we assume that p is an odd composite number. Now denote r as the smallest prime divisor of p ($r \neq 2$). We already know from the proof of **Theorem 1.1** that $\binom{p+r}{p} \not\equiv \lfloor \frac{p+r}{p} \rfloor \pmod{p}$.

Because $r \neq 2$ we know that $r+1$ is composite. Note that $p+r+1$ is relatively prime to p because every prime factor that would divide both would also divide $r+1$ and this prime factor would therefore be smaller than r which is a contradiction.

Now by **Lemma 2.1** we can deduce that

$$\binom{p+r+1}{p} \equiv \binom{p+r}{p} \pmod{p}$$

which means

$$\binom{p+r+1}{p} \not\equiv \lfloor \frac{p+r+1}{p} \rfloor \pmod{p}.$$

Because $p < p+r < p+r+1 < 2p$ and therefore $\lfloor \frac{p+r}{p} \rfloor = \lfloor \frac{p+r+1}{p} \rfloor \pmod{p}$.

Now by **Theorem 1.4** we know that for every natural n in the sequence $c_k = k \cdot l(p) + (p+r+1)$ we have $\binom{n}{p} \not\equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$.

Since p and $l(p)$ have the same prime factors due to the formula for $l(p)$ so $p+r+1$ which is relatively prime to p is also relatively prime to $l(p)$.

Again by **Theorem 2.3** we know there exists a prime number in the sequence c_k as defined above. Therefore there exists a prime number q such that

$$\binom{q}{p} \not\equiv \lfloor \frac{q}{p} \rfloor \pmod{p}.$$

This completes the rest of the proof. □

3. CONCLUSION

We see here, that a simple Olympiad problem has lead us to all the above results mentioned in this paper. This is just another testament of the beauty and originality of Olympiad mathematics.

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